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Exact *n*th Derivatives of Eigenvalues and Eigenvectors

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Computing derivatives of eigenvalues and eigenvectors is of considerable importance in mathematics, physics, and engineering. These derivatives are essential for sensitivity analysis, which is used to study the effect of change in various parameters of an eigensystem on its performance. General solutions of eigenfunctions' derivatives have not been available. In this paper, exact analytical solutions for the *n*th derivatives of unrepeated eigenvalues and corresponding eigenvectors are given for general nonlinear and linear eigenvalue problems. The application of the theory is illustrated with examples.

I. Introduction

THE noted mathematician Richard Bellman[1] called the problem of finding derivatives of eigenfunctions "a problem of great theoretical and practical interest." This interest has created a very active area of research in the past 150 years. One of the classic works on perturbation theory of linear operators, which includes the theory of the derivatives of eigenvalue and eigenvectors for the linear eigenvalue problem, was written by Kato.[2] In 1846 Jacobi[3] derived his result for the first simple eigenvalue derivative for a linear eigenvalue problem. Hundreds of papers have been published since, most of them about the first and second derivatives of eigenfunctions (eigenvalues and eigenvectors) for linear systems and their application. All of them expounded either analytical or numerical methods, and there has been much duplication of effort and result. For example, Ref. 4 reported results obtained over 10 years earlier.[5]

The topic of eigenfunctions' derivatives remains of considerable interest as indicated by many recent papers,[6-12] but complete analytical solutions for the nonlinear eigenvalue problem have eluded researchers. Jankovic[13-15] presented for the first time analytical solutions, valid under certain conditions, for the *n*th eigenfunctions' derivatives for a nonlinear eigenvalue problem.

Using these solutions, he also found the exact solution for the critical load of a cantilever column subjected to a compressive follower force. This nonlinear, nonconservative problem was discussed most recently by Chen and Ku,[16] who developed an approximate solution for the eigenvalue sensitivity and critical load using the finite element method.

Murthy and Haftka[17] and Adelman and Haftka[18] presented excellent surveys of the most important results on the eigenfunctions' derivatives and listed over 150 references to which the reader is referred for more information.

The nonlinear eigenvalue problem is defined here by a matrix equation

$$A(\zeta, \lambda)x = 0 \tag{1}$$

and normalization condition

$$x^\dagger K(\zeta, \lambda)x = 1 \tag{2}$$

where A is an $m \times m$ complex matrix, ζ a complex variable, λ a simple eigenvalue of A, x the corresponding eigenvector, and $x^\dagger$ its complex conjugate transpose; the eigenvector is K normalized by a nonsingular positive definite Hermitian K. Elements of A and K are analytic and, in general, nonlinear functions of ζ and λ and possibly other parameters. The purpose of this work is to present the exact, general analytical solutions for the nth derivatives of eigenvalues and eigenvectors, defined by Eqs. (1) and (2), with respect to a complex variable ζ.

No previous work has given exact analytical solutions of this general problem.

The eigenvalues can be obtained by setting the determinant of A to zero, $\det A = 0$, for various values of ζ and solving the resulting characteristic equation for λ. It should be emphasized that K normalization is not unique, because it determines eigenvector x only within an arbitrary constant; this apparent ambiguity is of no importance, as pointed out in Ref. 17. However, the nth derivative $x^{(n)}$ will be unique in the sense that it will always be the derivative of the chosen eigenvector x, regardless of the method used to determine the arbitrary constant. One of these methods[17] requires an element of the eigenvector to be set to one.

This paper will proceed as follows. The fundamental equations relating the nth derivatives of eigenfunctions are derived first [Eqs. (5) and (11)]. These solutions lead to five important theorems that are valid for both linear and nonlinear eigenvalue problems and are proved in the Appendix.

Theorem 1 and 3 give general solutions for the nth derivative of eigenvalue $\lambda^{(n)}$, using left eigenvector y and without using left eigenvector y, respectively. Theorem 2 gives the solution for $\lambda^{(n)}$ when A is K Hermitian. Theorem 4 presents a general solution for the nth derivative of eigenvector $x^{(n)}$. Theorem 5 gives solutions for the nth derivatives of eigenvalue and eigenvector if $\alpha = x^\dagger K_\lambda x \neq 0$, where K_λ is a partial derivative of matrix K with respect to eigenvalue λ. Lemma introduces a special matrix $Q = (A + xx^\dagger K)^{-1}$ with very important properties and is used in Theorems 2 and 4. The theory is then applied to three linear examples[19-22] and one nonlinear[13] example.

Fundamental Equations

The nth derivative of Eq. (1) with respect to the parameter ζ is given by

$$(Ax)^{(n)} = 0$$

or using the Leibnitz rule

$$A^{(n)}x + Ax^{(n)} + \sum_{k=1}^{n-1} \binom{n}{k} A^{(n-k)}x^{(k)} = 0 \tag{3}$$

Since

$$A^{(1)} = A_\zeta + A_\lambda \lambda^{(1)}$$

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where A_ζ and A_λ are partial derivatives with respect to ζ and λ, respectively, then

$$A^{(n+1)} = A_\zeta^{(n)} + (A_\lambda \lambda^{(1)})^{(n)}$$

$$= A_\zeta^{(n)} + A_\lambda \lambda^{(n+1)} + \sum_{k=0}^{n-1} \binom{n}{k} A_\lambda^{(n-k)} \lambda^{(k+1)}$$

or

$$A^{(n)} = A_\zeta^{(n-1)} + A_\lambda \lambda^{(n)} + \sum_{k=0}^{n-2} \binom{n-1}{k} A_\lambda^{(n-k-1)} \lambda^{(k+1)} \quad (4)$$

Substitution of Eq. (4) into Eq. (3) leads to

$$Ax^{(n)} + A_\lambda x \lambda^{(n)} = -z_n \quad (5)$$

where matrix z_n is given by

$$z_n = A_\zeta^{(n-1)} x + \sum_{k=1}^{n-1} \binom{n}{k} A^{(n-k)} x^{(k)}$$

$$+ \sum_{k=0}^{n-2} \binom{n-1}{k} A_\lambda^{(n-k-1)} x \lambda^{(k+1)} \quad (6)$$

and $x^{(n-1)}$ is the highest derivatives of x on the right-hand side.

Equation (6) will be evaluated later for the first three derivatives for the nonlinear [Eq. (18)], the general linear [Eq. (21)], and the linear eigenvalue problem [Eq. (22)]. Differentiating of Eq. (2) using Leibnitz rule leads to

$$(x^\dagger K x)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{\dagger(n-k)} (Kx)^{(k)}$$

$$= 0$$

where

$$(Kx)^{(k)} = \sum_{m=0}^{k} \binom{k}{m} K^{(k-m)} x^{(m)}$$

Thus,

$$(x^\dagger K x)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{\dagger(n-k)} \sum_{m=0}^{k} \binom{k}{m} K^{(k-m)} x^{(m)}$$

$$= x^{\dagger(n)} K x + \sum_{k=1}^{n} \binom{n}{k} x^{\dagger(n-k)}$$

$$\times \left[K^{(k)} x + K x^{(k)} + \sum_{m=1}^{k-1} \binom{k}{m} K^{(k-m)} x^{(m)} \right]$$

or

$$(x^\dagger K x)^{(n)} = x^{\dagger(n)} K x + x^\dagger K x^{(n)} + x^\dagger K^{(n)} x$$

$$+ \sum_{m=1}^{n-1} \binom{n}{m} [x^\dagger K^{(n-m)} x^{(m)} + x^{\dagger(n-m)} K^{(m)} x$$

$$+ x^{\dagger(n-m)} K x^{(m)}] + \sum_{k=1}^{n-1} \binom{n}{k} x^{\dagger(n-k)} \sum_{m=1}^{k-1} \binom{k}{m} K^{(k-m)} x^{(m)}$$

$$= 0$$

By substituting

$$K^{(n)} = K_\zeta^{(n-1)} + K_\lambda \lambda^{(n)} + \sum_{k=0}^{n-2} \binom{n-1}{k} K_\lambda^{(n-k-1)} \lambda^{(k+1)}$$

into the preceding equation one gets

$$x^{\dagger(n)} K x + x^\dagger K x^{(n)} = -\eta_n - \alpha \lambda^{(n)} \quad (7)$$

where scalars α and η_n are given by

$$\alpha = x^\dagger K_\lambda x \quad (8)$$

and

$$\eta_n = x^\dagger K_\zeta^{(n-1)} x + \sum_{k=0}^{n-2} \binom{n-1}{k} x^\dagger K_\lambda^{(n-k-1)} x \lambda^{(k+1)}$$

$$+ \sum_{m=1}^{n-1} \binom{n}{m} [x^\dagger K^{(n-m)} x^{(m)} + x^{\dagger(n-m)} K^{(m)} x$$

$$+ x^{\dagger(n-m)} K x^{(m)}] + \sum_{k+}^{n-1} \binom{n}{k} x^{\dagger(n-k)} \sum_{m=1}^{k-1} \binom{k}{m} K^{(k-m)} x^{(m)} \quad (9)$$

Equations (5) and (7) are fundamental equations with the unknowns $\lambda^{(n)}$ and $x^{(n)}$. We observe that the left-hand side of Eq. (7) must be real because it is a sum of a complex scalar $x^{\dagger(n)} K x$ and its complex conjugate $x^\dagger K x^{(n)}$. Therefore, its right-hand side also must be a real scalar number.

Both of the fundamental equations cannot be solved readily because in addition to unknowns $\lambda^{(n)}$ and $x^{(n)}$, the unknown $x^{\dagger(n)}$ also appears in Eq. (7). We will resolve this problem by first considering the case for which

$$x^{\dagger(n)} K x = (x^\dagger K x^{(n)})^\dagger = x^\dagger K x^{(n)} \quad (10)$$

This is true only under the following conditions:

$$x^{\dagger(n)} K x \text{ is real} \quad (11a)$$

$$K \text{ is diagonal matrix} \quad (11b)$$

The condition (11a) is obvious and the condition (11b) can be easily verified by writing

$$x^{(n)} = u^{(n)} + jv^{(n)}; \qquad x^{\dagger(n)} = u^{(n)} - jv^{(n)}$$

and carrying out the matrix multiplication in Eq. (10).

The condition (11b) can always be fulfilled because the nonsingular positive definite Hermitian K can always be diagonalized by similarity transformation[23] and is given by

$$K = S^\dagger \text{diag}(\omega_1, \omega_2, \ldots, \omega_m) S$$

where $\text{diag}(\omega_1, \omega_2, \ldots, \omega_m)$ is a diagonal $m \times m$ matrix with real and positive eigenvalues of K for the diagonal elements, and

$$S = [s_1, s_2, \ldots, s_m]$$

where $s_1, s_2, \ldots, s_m$ are eigenvectors of K. Therefore, we can use a linear transformation

$$w = Sx \quad (12)$$

and write Eqs. (1) and (2) as

$$\mathbf{A} w = 0 \quad (13)$$

$$w^\dagger w = 1 \quad (14)$$

where

$$\mathbf{A} = A(\zeta, \lambda) S^\dagger \quad (15)$$

The eigenvalues of $\mathbf{A}$ are the same as the eigenvalues of A because they are found from

$$\det \mathbf{A} = \det A(\zeta, \lambda) S\dagger$$
$$= \det A(\zeta, \lambda) \det S\dagger$$
$$= 0$$

and $\det S\dagger \neq 0$.

From Eq. (12), $x = S\dagger w$ and the derivatives of x can be obtained by direct differentiation as $x^{(n)} = (S\dagger w)^{(n)}$.

Solutions for $\lambda^{(n)}$ (Theorems 1-3) will be obtained directly from Eq. (5), and Eq. (10) will not be required (see Appendix). However, it will be required for calculating $x^{(n)}$ in Theorem 4 and for calculating both $x^{(n)}$ and $\lambda^{(n)}$ in Theorem 5, as explained in the Appendix.

We can make an important conclusion here. There is no loss of generality by solving the fundamental system of equations under the condition given by Eq. (10) because condition (11b) is always fulfilled for the transformed Eqs. (13) and (14). Namely,

$$w\dagger^{(n)} w = w\dagger w^{(n)} \tag{16}$$

It should be noted that in this case all the derivatives of $K = I$ will be equal to zero and Eq. (19) will be much simpler to evaluate, but it will be more complicated to evaluate Eq. (18).

The fundamental system of equations under the condition given by Eq. (10) now becomes

$$A x^{(n)} + A_\lambda x \lambda^{(n)} = -z_n \tag{5}$$

$$x\dagger K x^{(n)} = -\tfrac{1}{2}(\eta_n + \alpha\lambda^{(n)}) \tag{17}$$

where z_n and μ_n are given by Eqs. (6) and (9). They are evaluated here for the first three derivatives, namely, for $n = 1$, 2, and 3. Thus,

$$z_1 = A_\zeta x$$
$$z_2 = (A_\zeta^{(1)} + \lambda^{(1)} A_\lambda^{(1)}) x + 2A^{(1)} x^{(1)} \tag{18}$$
$$z_3 = (A_\zeta^{(2)} + \lambda^{(1)} A_\lambda^{(2)} + 2\lambda^{(2)} A_\lambda^{(1)}) x + 3A x^{(1)} + 3A^{(1)} x^{(2)}$$

and

$$\eta_1 = x\dagger K_\zeta x$$
$$\eta_2 = x\dagger K_\zeta^{(1)} x + x\dagger K_\lambda^{(1)} x \lambda^{(1)}$$
$$+ 2[x\dagger K^{(1)} x^{(1)} + x\dagger^{(1)} K^{(1)} x + x\dagger^{(1)} K x^{(1)}]$$
$$\eta_3 = x\dagger K_\zeta^{(2)} x + x\dagger K_\lambda^{(2)} x \lambda^{(1)} + 2x\dagger K_\lambda^{(1)} x \lambda^{(2)} \tag{19}$$
$$+ 6x\dagger^{(1)} K^{(1)} x^{(1)} + 3[x\dagger K^{(2)} x^{(1)} + x\dagger^{(2)} K^{(1)} x$$
$$+ x\dagger^{(2)} K x^{(1)}] + 3[x\dagger K^{(1)} x^{(2)} + x\dagger^{(1)} K^{(1)} x + x\dagger^{(1)} K x^{(2)}]$$

For the general linear eigenvalue problem given by

$$(\mathbf{A} - \lambda \mathbf{B}) x = 0 \tag{20}$$

the following equations are obtained instead of Eqs. (18), respectively:

$$z_1 = (\mathbf{A}_\zeta - \lambda \mathbf{B}_\zeta) x$$
$$z_2 = (\mathbf{A}_{\zeta\zeta} - 2\lambda^{(1)} \mathbf{B}_\zeta - \lambda \mathbf{B}_{\zeta\zeta}) x + 2(\mathbf{A}_\zeta - \lambda^{(1)} \mathbf{B} - \lambda \mathbf{B}_\zeta) x^{(1)}$$
$$z_3 = (\mathbf{A}_{\zeta\zeta\zeta} - 3\lambda^{(2)} \mathbf{B}_\zeta - 3\lambda^{(1)} \mathbf{B}_{\zeta\zeta} - \lambda \mathbf{B}_{\zeta\zeta\zeta}) x \tag{21}$$
$$+ 3(\mathbf{A}_{\zeta\zeta} - \lambda^{(2)} \mathbf{B} - 2\lambda^{(1)} \mathbf{B}_\zeta - \lambda \mathbf{B}_{\zeta\zeta}) x^{(1)}$$
$$+ 3(\mathbf{A}_\zeta - \lambda^{(1)} \mathbf{B} - \lambda \mathbf{B}_\zeta) x^{(2)}$$

For linear eigenvalue problem for which $\mathbf{B} = I$, we have

$$z_1 = \mathbf{A}_\zeta x$$
$$z_2 = \mathbf{A}_{\zeta\zeta} x + 2(\mathbf{A}_\zeta - \lambda^{(1)} I) x^{(1)} \tag{22}$$
$$z_3 = \mathbf{A}_{\zeta\zeta\zeta} x + 3(\mathbf{A}_{\zeta\zeta} - \lambda^{(2)} I) x^{(1)} + 3(\mathbf{A}_\zeta - \lambda^{(1)} I) x^{(2)}$$

Solutions of the general system of Eqs. (5) and (17) are now discussed. The general solution for $\lambda^{(n)}$, given in Theorem 3, and the solution for $x^{(n)}$, given in Theorem 4, are

$$x^{(n)} = -\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n) x - Q(A_\lambda x \lambda^{(n)} + z_n) \tag{23}$$

$$\lambda^{(n)} = -\frac{x\dagger \mathbf{K} z_n}{x\dagger \mathbf{K} A_\lambda x} \tag{24}$$

respectively, where $Q = (A + xx\dagger K)^{-1}$ is a nonsingular matrix, $\mathbf{K} = KQ$, and $x\dagger \mathbf{K} A_\lambda x \neq 0$. They are now shown to satisfy Eqs. (5) and (17). By premultiplying Eq. (5) from the left by Q and using Eq. (23) we get

$$QA[-\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n) x - Q(A_\lambda x \lambda^{(n)} + z_n)]$$
$$+ Q(A_\lambda x^{(n)} + z_n) = 0$$

or

$$(I - QA) Q (A_\lambda x \lambda^{(n)} + z_n) = 0$$

However,

$$I - QA = xx\dagger K$$

because

$$QQ^{-1} = Q(A + xx\dagger K)$$
$$= I$$

and the equation becomes

$$x(x\dagger K Q A_\lambda x \lambda^{(n)} + x\dagger K Q z_n) = 0$$

which is satisfied because of Eq. (24). Similarly after premultiplying Eq. (23) from the left by $x\dagger K$ it becomes

$$x\dagger K x^{(n)} = -\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n) x\dagger K x - x\dagger K Q (A_\lambda x \lambda^{(n)} + z_n)$$

and after using Eqs. (2) and (24), we obtain

$$x\dagger K x^{(n)} = -\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n)$$

This completes the proof. Next, the theory will be applied to three linear examples and one nonlinear example.

Example 1

This is a linear example taken from Greensite.[19] It is required that the shift in the complex eigenvalue λ_2 be calculated when element a_{23} of matrix $\mathbf{A}$ changes from 1 to 0.2 where

$$\mathbf{A} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \tag{25}$$

and $Ax = \mathbf{0}$, for $A \equiv \mathbf{A} - \lambda I$.

The eigenvalues of the unperturbed matrix (25), indicated by superscript 0, are

$$\lambda_1^0 = 1, \qquad \lambda_2^0 = -1 + j, \qquad \lambda_3^0 = -1 - j, \qquad j = \sqrt{-1}$$

and the corresponding eigenvectors and left eigenvectors are

$$x_1 = \frac{1}{\sqrt{55}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad x_2 = \frac{1}{\sqrt{55}}\begin{pmatrix} 5 \\ -(3+4j) \\ 2+j \end{pmatrix}$$

$$x_3 = \frac{1}{\sqrt{55}}\begin{pmatrix} 5 \\ -(3-4j) \\ 2-j \end{pmatrix}$$

$$y_1 = \frac{\sqrt{55}}{5}\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}, \quad y_2 = \frac{\sqrt{55}}{10}\begin{pmatrix} 1-j \\ -j \\ j \end{pmatrix}$$

$$y_3 = \frac{\sqrt{55}}{10}\begin{pmatrix} 1+j \\ j \\ -j \end{pmatrix}$$

respectively, where $y_i^\dagger x_i = 1$ and $x_i^\dagger x_i = 1$ for $i = 1, 2, 3$. Then, using Eq. (A1), Theorem 1, and the first of Eqs. (22), the first derivative of the second eigenvalue with respect to $\zeta = a_{23}$ is found to be

$$\lambda_2^{(1)} = y_2^\dagger \mathbf{A}_\zeta x_2$$

where

$$\mathbf{A}_\zeta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$\lambda_2^{(1)} = -0.1 + 0.2j$$

Since the shift in the second eigenvalue λ_2^0 is $\Delta = -0.8$, the first approximation of λ_2 is

$$\hat{\lambda}_2 = \lambda_2^0 + \Delta\lambda_2^{(1)}$$
$$= -0.92 + 0.84j$$

which is the same as in Ref. (19).

The first derivative of the second eigenvector is obtained from Eq. (A13), Theorem 4, as

$$x_2^{(1)} = x_2\lambda_2^{(1)} - QA_\zeta x_2$$

$$= \begin{pmatrix} -0.03187126622554 - 0.05271017106531j \\ 0.01470981518102 - 0.07232325797333j \\ -0.03187126622554 - 0.02206472277152j \end{pmatrix}$$

The second eigenvalue derivative is obtained from Eq. (A1)

$$\lambda_2^{(2)} = y_2^\dagger z_2$$

where, from Eq. (22),

$$z_2 = 2(\mathbf{A}_\zeta - \lambda^{(1)} I)x_2^{(1)}$$

Therefore,

$$\lambda_2^{(2)} = 0.032 - 0.074j$$

and the second-order approximation using Taylor's series expansion is

$$\hat{\lambda}_2 = \lambda_2^0 + \Delta\lambda_2^{(1)} + \frac{\Delta^2}{2}\lambda_2^{(2)}$$
$$= -0.90976 + 0.81632j$$

The third eigenvalue derivative is also obtained from Eq. (A1)

$$\lambda_2^{(3)} = y_2^\dagger z_3$$

where, from Eq. (22)

$$z_3 = -3\lambda_2^{(2)} x_2^{(1)} + 3(\mathbf{A}_\zeta - \lambda^{(1)} I)x_2^{(2)}$$

and from Eq. (A13)

$$x_2^{(2)} = -(x_2^{\dagger(1)} x_2^{(1)})x_2 + x_2\lambda_2^{(2)} - Qz_2$$

$$= \begin{pmatrix} -0.00077783719669 + 0.02678523618870j \\ -0.00881920282898 + 0.03381920235708j \\ 0.01265044105567 + 0.00772933924966j \end{pmatrix}$$

The third eigenvector derivative is also obtained from Eq. (A13) as

$$x_2^{(3)} = -3(x_2^{\dagger(1)} x_2^{(2)})x_2 + x_2\lambda_2^{(3)} - Qz_3$$

Thus,

$$\lambda_2^{(3)} = -0.02592 + 0.06744j$$

and the third-order, second eigenvalue approximation is

$$\hat{\lambda}_2 = \lambda_2^0 + \Delta\lambda_2^{(1)} + \frac{\Delta^2}{2}\lambda_2^{(2)} + \frac{\Delta^3}{6}\lambda_2^{(3)}$$
$$= 0.9075 + 0.8106j$$

which should be compared with the solution for λ_2 for the perturbed matrix, for which $\mathbf{A}(2,3) = 0.2$, obtained using MATLAB[24] software

$$\lambda_2 = -0.9067 + 0.8081j$$

Greensite[19] gives $\lambda_2 = = -0.9075 + 0.81j$ for the perturbed matrix. Compared to either value the third-order second eigenvalue approximation is excellent.

Similarly, the third eigenvector approximation is obtained from

$$\hat{x}_2 = x_2^0 + \Delta x_2^{(1)} + \frac{\Delta^2}{2}x_2^{(2)} + \frac{\Delta^3}{6}x_2^{(3)}$$

$$= \begin{pmatrix} 0.6988 + 0.0525j \\ -0.4199 - 0.4682j \\ 0.3001 + 0.1553j \end{pmatrix}$$

which gives $x_2^\dagger x_2 = 1.0008$ for the normalization condition, instead of 1 for the exact solution, where the error 0.0008 is due to truncating higher order terms in the Taylor's series expansion for x_2. The value for $\hat{x}_2$ compares well with the calculated value for x_2 of the perturbed matrix,

$$x_2 = \begin{pmatrix} 0.6742 \\ -0.4045 - 0.5394j \\ 0.2697 + 0.1348j \end{pmatrix}$$

The following example describing a damp vibration problem was discussed by Tan[21,22] to test iterative computational methods of calculating eigenfunctions' derivatives, and it had been used by many others as a benchmark. Note that the eigenvalues and corresponding eigenvectors in these references are misprinted. Corrected values, verified using MATLAB software, are given subsequently.

Example 2

Given the 8 × 8 matrix $A = A_1 + \sigma I_8$, where σ is a real shift parameter and A_1 is the 8 × 8 matrix given by

$$A_I = \begin{pmatrix} -C_1 & -D_1 \\ I_4 & O_4 \end{pmatrix}$$

$$C_1 = \begin{bmatrix} 3\alpha & -(1+\alpha^2+2\beta^2) & \alpha(1+2\beta^2) & -\beta^2(\alpha^2+\beta^2) \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -1+2\alpha^2 & \alpha-\alpha(\alpha^2+2\beta^2) & 2\alpha^2\beta^2 & -\alpha\beta^2(\alpha^2+\beta^2) \\ 2\alpha & -(\alpha^2+2\beta^2) & 2\alpha\beta^2 & -\beta^2(\alpha^2+\beta^2) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The identity matrix I_m is of dimension $m \times m$, where $m = 4$ or $m = 8$; O_4 is a 4 × 4 null matrix; α is a real parameter; and $\beta = \alpha + 1$. The eigenvalues of A are given by

$$\lambda_{(1,2)} = -\sigma \pm j, \qquad \lambda_{(3,4)} = -\sigma \pm \beta j, \qquad \lambda_{(5,6)} = 1 - \sigma \pm \beta j$$

$$\lambda_{(7)} = -\sigma, \qquad \lambda_{(8)} = 1 - \sigma - j$$

The eigenvector corresponding to λ_3 is

$$x_3 = (-\beta^4, -\beta^3 j, \beta^2, \beta j, \beta^3 j, -\beta^2, -\beta j, 1)^T$$

and $x_4 = \text{conj}(x_3)$.

The exact solution will be found for the first and second derivatives of λ_3 and its corresponding normalized eigenvector x_3 with respect to the parameter β. They are compared subsequently with the already given exact closed-form solutions.

The normalized eigenvector x_3 is

$$\bar{x}_3 = (1/d)x_3$$

where

$$d = (\beta^8 + 2\beta^6 + 2\beta^4 + 2\beta^2 + 1)^{0.5}$$

so that $x_3^\dagger x_3 = 1$.

The third eigenvectors' and eigenvalues' derivatives were calculated with full machine accuracy of 15 significant digits for $\beta \in (1, 4)$, using Theorems 1, 3, and 4. The computer used was Macintosh Quadra 700 and MATLAB software. None of the effects associated with ill-conditioning were present, such as poor accuracy, lack of convergence, and other problems and restrictions usually associated with various numerical techniques.[20,22]

For $\beta = 1$, the multiple eigenvalues occur, and the theory presented here does not apply. It is remarkable, however, that excellent accuracy was obtained for the ill-conditioned problem of $\beta \approx 1$, for which $\lambda_1 \approx \lambda_3 \approx \lambda_5$ and $\lambda_2 \approx \lambda_4 \approx \lambda_6$. For example, for $\beta = 1 + 10^{-6}$ the first derivative's accuracy was 11 significant digits and the second derivative's accuracy was six significant digits. The error is due entirely to the fact that matrix Q is singular for multiple eigenvalues and is ill conditioned for very close eigenvectors.

The plots of the derivatives of the real and imaginary parts of x_3 are shown in Figs. 1–8.

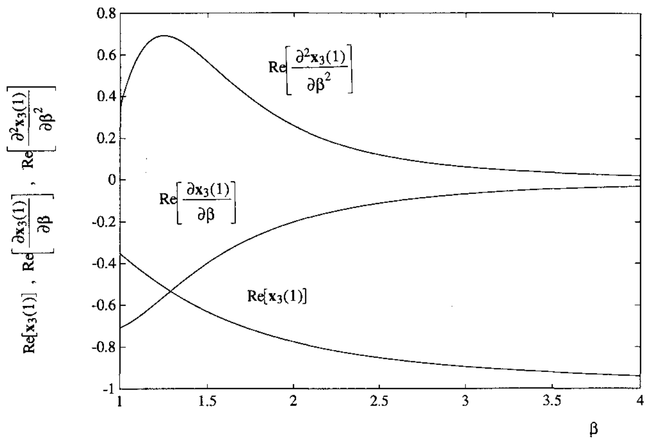

Fig. 1 Real part of $x_3(1)$ and its first and second derivatives with respect to a real parameter β.

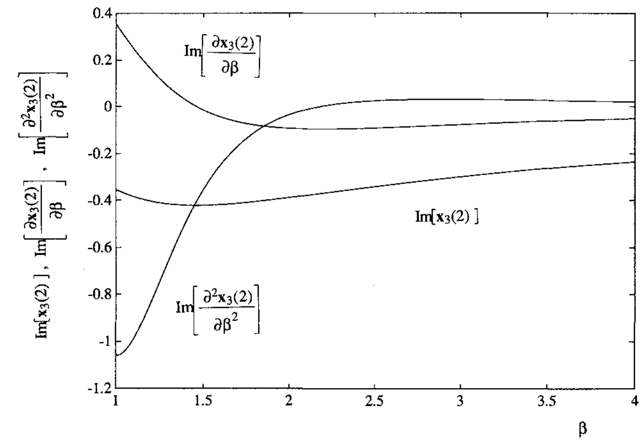

Fig. 2 Imaginary part of $x_3(2)$ and its first and second derivatives with respect to a real parameter β.

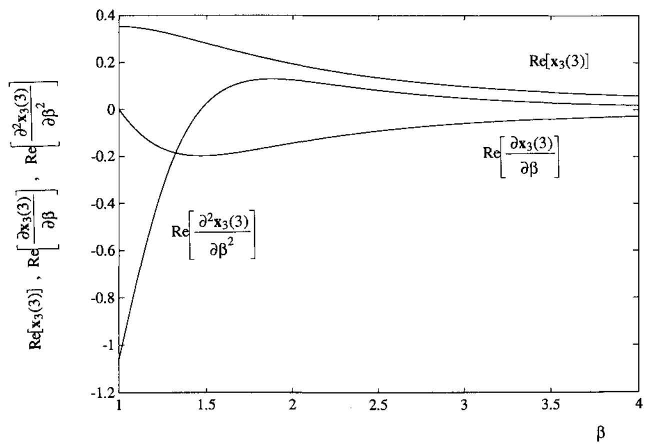

Fig. 3 Real part of $x_3(3)$ and its first and second derivatives with respect to a real parameter β.

The following example is a well-known problem of finding eigenfunctions' derivatives for the Hessenberg[20] matrix. This was also discussed by Tan,[21,22] where iterative computational methods were used.

Example 3

The Hessenberg matrix $\boldsymbol{H}$ of the nth order

$$\boldsymbol{H} = \begin{bmatrix} \alpha & 1 & 0 & \cdot & \cdot & 0 \\ \alpha^2 & \alpha & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \alpha^n & \alpha^{n-1} & \cdot & \cdot & \cdot & \alpha \end{bmatrix}$$

has $m = [n/2]$ multiple eigenvalues equal to 0 and the remaining $n - m$ simple eigenvalues are

$$\lambda_k = 4\alpha \cos^2 \frac{k\pi}{n+2}, \qquad k = 1, 2, \ldots, n - m$$

where α is a parameter, and $[n/2]$ denotes the largest integer not exceeding $n/2$. The corresponding jth element of the kth eigenvector is

$$x_{k,j} = \alpha^{j-1} 2^{j-2} (\cos \varphi)^{j-2} \frac{\sin (j+1)\varphi}{\sin \varphi} \qquad j = 1, \ldots, n$$

where $\varphi = [k\pi/(n+2)]$ (Ref. 22).

It is required to find the first and the second derivatives of the simple eigenvalues and eigenvectors of the Hessenberg matrix.

This is a difficult problem for large matrices and is a standard benchmark for comparison with analytical closed-form solutions. Tan[21,22] used iterative numerical methods to obtain the first derivatives of the first eigenvalue and eigenvector for matrices with dimensions $n \leq 25$ and discussed their efficiencies. The largest matrix for which he could obtain 14 digits accuracy was $n = 16$ for $\alpha = 0.1$, 1.0, 3.0. He also found ill conditioning for small α, $0.0001 \leq \alpha \leq 5$, with $n = 12$ and for $0.25 \leq \alpha \leq 2.5$ with $n = 25$, for which cases the simple eigenvalues λ_k ($k = 1, 2, \ldots, n - m$) are close together. He concluded that the ill conditioning was worse for smaller α and larger n, which resulted in reduced accuracy and an increased number of iterations.

Using Theorems 1 and 4, the first two eigenvectors' derivatives of the first and second eigenvalues were calculated for all of the preceding cases[21,22] and compared with the closed-form exact analytical solutions. Full machine accuracy of 14 significant digits was obtained as expected. Also for a large matrix, $n = 100$ and complex $\alpha = 0.1*i$, the first derivatives were obtained with the same 14 digits accuracy. Furthermore, a remarkable accuracy of 10–14 significant digits for the first two derivatives of the first two eigenvalues and eigenvectors was obtained for $\alpha = 0.25$ and a very large $\boldsymbol{H}$ matrix of dimension $n = 250$. No numerical or analytical method was ever reported in the literature capable of solving this type of problem.

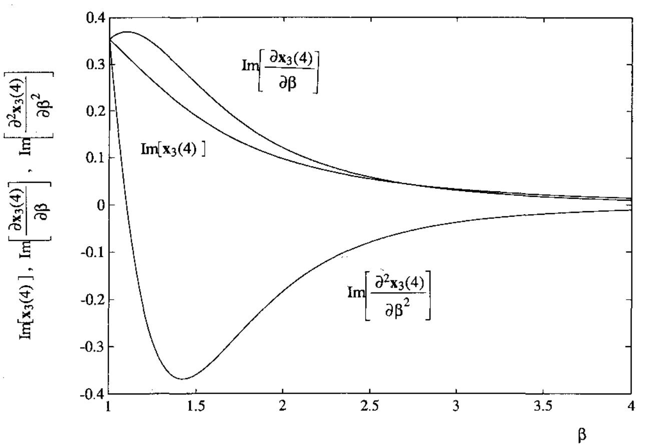

Fig. 4 Imaginary part of $x_3(4)$ and its first and second derivatives with respect to a real parameter β.

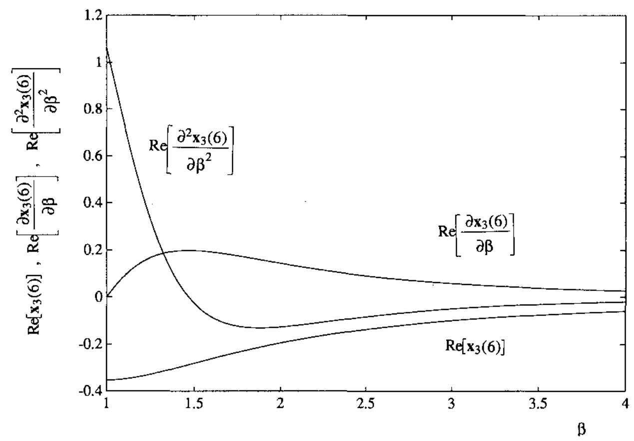

Fig. 6 Real part of $x_3(6)$ and its first and second derivatives with respect to a real parameter β.

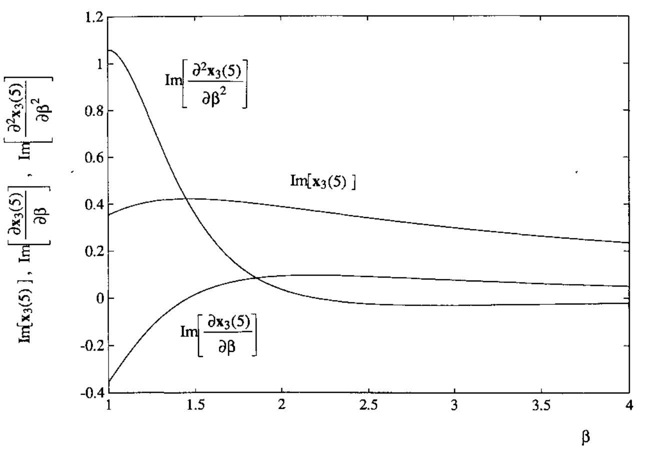

Fig. 5 Imaginary part of $x_3(5)$ and its first and second derivatives with respect to a real parameter β.

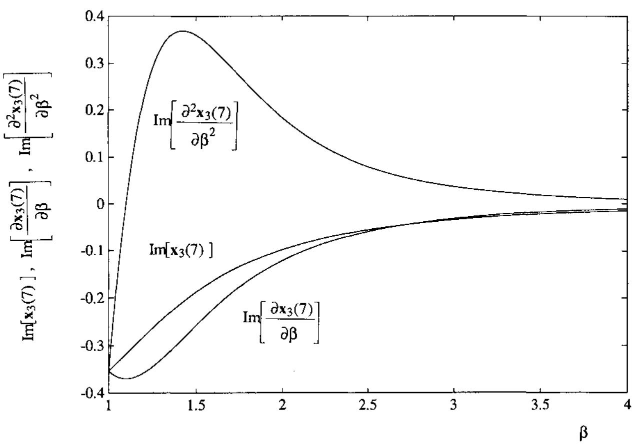

Fig. 7 Imaginary part of $x_3(7)$ and its first and second derivatives with respect to a real parameter β.

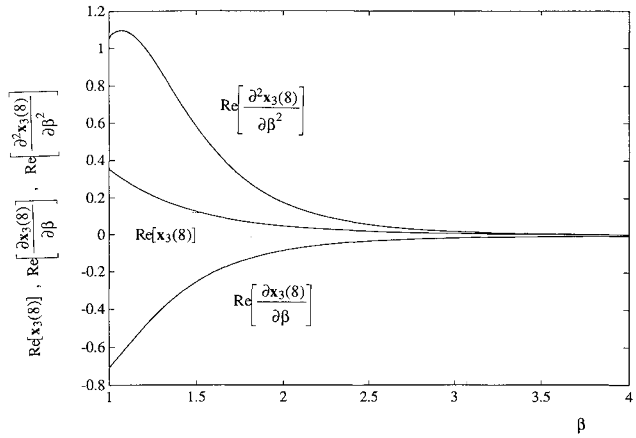

Fig. 8 Real part of $x_3(8)$ and its first and second derivatives with respect to a real parameter β.

Example 4

Find the first derivatives of eigenvalues and eigenvectors describing a nonconservative nonlinear eigenvalue problem of a cantilever column subjected to a compressive follower force.

This problem was solved previously[13] using the formulas identical to those obtained from Theorem 5 for the first derivatives. It is solved here using Theorems 1 and 4 and identical results were obtained. Using the first eigenvalue derivative, the solution was also obtained for the critical load as

$$P_{cr} = 20.05095361897375\ldots$$

for which the critical frequency is

$$\omega_{cr} = 11.01555764092285\ldots$$

Conclusions

Exact analytical solutions for the nth derivatives of unrepeated eigenvalues and eigenvectors for a general nonlinear eigenvalue problem were presented here for the first time. The solutions are also valid for general linear eigenvalue problems. Five theorems that give these explicit solutions were stated and proved. The theory was applied to three linear examples that show full agreement with closed-form exact solutions. The first derivatives of eigenfunctions describing a nonconservative nonlinear problem of cantilever column subjected to a compressive follower force were calculated; the critical load was also calculated to the maximum machine accuracy of 14 significant digits.

Appendix: Proofs of the Theorems

Strictly speaking, Theorems 1, 2, and 3 that give the nth derivatives of eigenvalue λ are not derived under the condition given by Eq. (10), and their validity is not restricted by it. However, the higher order derivatives $\lambda^{(n)} = \lambda(x^{(n-1)}, x^{(n-2)}, \ldots, x)$ given by these theorems require the derivatives of eigenvector up to the order of $(n-1)$, which are calculated in Theorem 4 using Eq. (10). If $(x^{(i-1)}, i = 1, 2, \ldots, n)$ are available from elsewhere and are not obtained using Eq. (10), Theorems 1–3 are still valid. Theorem 5 was derived using Eq. (10).

The following theorem requires the left eigenvector $y\dagger$ for calculation of the derivative $\lambda^{(n)}$.

Theorem 1. For the nonlinear eigenvalue problem, the nth derivative of the unrepeated eigenvalue λ is

$$\lambda^{(n)} = -\frac{y\dagger z_n}{y\dagger A_\lambda x} \tag{A1}$$

where $y\dagger$ is the left eigenvector, and $y\dagger A_\lambda x \neq 0$.

Proof. Premultiplying Eq. (5) by $y\dagger$ gives

$$y\dagger A x^{(n)} + y\dagger A_\lambda x \lambda^{(n)} = -y\dagger z_n$$

and, since by definition $y\dagger A = 0$, the solution for $\lambda^{(n)}$ follows directly.

For the general linear eigenvalue problem given by Eq. (20), this theorem gives, for $\lambda^{(n)}$,

$$\lambda^{(n)} = \frac{y\dagger z_n}{y\dagger \mathbf{B} x} \tag{A2}$$

because in this case $A_\lambda = -\mathbf{B}$.

For $\mathbf{B} = I$ the result is

$$\lambda^{(n)} = y\dagger z_n \tag{A3}$$

because $y\dagger x = 1$. For nonlinear case z_n is given for the first three derivatives by Eq. (18) and for general linear and linear case by Eqs. (21) and (22), respectively. For example, using Eq. (21) for z_1 and z_2 and Eq. (A3), we get for the first and second derivatives

$$\lambda^{(1)} = y\dagger(\mathbf{A}_\xi - \lambda \mathbf{B}_\xi)x \tag{A4}$$

$$\lambda^{(2)} = y\dagger[(\mathbf{A}_{\xi\xi} - 2\lambda^{(1)}\mathbf{B}_\xi - \lambda\mathbf{B}_{\xi\xi})x + 2(\mathbf{A}_\xi - \lambda^{(1)}\mathbf{B} - \lambda\mathbf{B}_\xi)x^{(1)} \tag{A5}$$

respectively. Equation (A4) is a well-known result.

The following theorem determines $\lambda^{(n)}$, for which A is K Hermitian.

Theorem 2. For the nonlinear eigenvalue problem for which A is K Hermitian, $KA = A\dagger K$, the nth derivative of the unrepeated eigenvalue λ is

$$\lambda^{(n)} = -\frac{x\dagger K z_n}{x\dagger K A_\lambda x} \tag{A6}$$

where $x\dagger K A_\lambda x \neq 0$.

Proof. Premultiplying Eq. (5) by $x\dagger K \neq 0$ leads to Eq. (A6). If $K = I$, then

$$\lambda^{(n)} = -\frac{x\dagger z_n}{x\dagger A_\lambda x} \tag{A7}$$

which for the general linear eigenvalue problem, Eq. (20), becomes

$$\lambda^{(n)} = x\dagger z_n \tag{A8}$$

for the normalization condition $x\dagger\mathbf{B}x = 1$. For example, using this theorem and Eq. (21) for z_1 we obtain for the first derivative

$$\lambda^{(1)} = x\dagger(\mathbf{A}_\xi - \lambda\mathbf{B}_\xi)x$$

which is a well-known solution; for $\mathbf{B} = I$ the result given by Bellman[1] is obtained.

A very important matrix Q and its properties is introduced next.

Lemma. Given the nonlinear eigenvalue problem $Ax = 0$ and normalization condition $x\dagger Kx = 1$, where K is a positive definite Hermitian and $Q = (A + xx\dagger K)^{-1}$ is a nonsingular matrix, a positive definite matrix $\mathbf{K} = KQ$ transforms the matrix A into $\mathbf{K}$ Hermitian $\underline{A} = \mathbf{K}A$ so that $\underline{A} = \underline{A}\dagger$. Furthermore, $\mathbf{K}$ is Hermitian if and only if A is K Hermitian.

Proof. First we prove that Q is nonsingular. Indeed, if Q is singular, that is, det $Q = 0$, then there is a nontrivial solution $x \neq 0$ of the system of linear equations

$$Qx = 0 \tag{A9}$$

However, from definition $Q = (A + xx^\dagger K)^{-1}$, it follows that $Q^{-1}x = x$, or

$$Qx = x \tag{A10}$$

which contradicts Eq. (A10). Therefore, it must be that det $Q \neq 0$, Q is nonsingular, and Eq. (A10) is satisfied if and only if $x = \mathbf{0}$. Since

$$Q(A + xx^\dagger K) = I \qquad (A^\dagger + Kxx^\dagger)Q^\dagger = I$$

by premultiplying the former and postmultiplying the later equation, we obtain from the equality of the left-hand sides

$$KQ(A + xx^\dagger K) = (A^\dagger + Kxx^\dagger)Q^\dagger K \tag{A11}$$

However,

$$KQxx^\dagger K = Kxx^\dagger Q^\dagger K$$

because of Eq. (A10), which reduces the preceding equation to $Kxx^\dagger K = Kxx^\dagger K$. Therefore, Eq. (A11) becomes

$$KQA = A^\dagger Q^\dagger K$$

or $\mathbf{K}A = A^\dagger \mathbf{K}^\dagger$, which is $\underline{A} = \underline{A}^\dagger$. A proof that $\mathbf{K}$ is Hermitian if and only if A is K Hermitian follows. Indeed, if A is K Hermitian, then $KA = A^\dagger K$ or

$$(A^\dagger + Kxx^\dagger)K = K(A + xx^\dagger K)$$

$$K(A + xx^\dagger K)^{-1} = (A^\dagger + Kxx^\dagger)^{-1}K$$

which is $KQ = Q^\dagger K$ or $\mathbf{K} = \mathbf{K}^\dagger$ and the proof is completed. The normalization condition is $x^\dagger \mathbf{K} x = 1$, which is the same as Eq. (2), because of Eq. (A10).

Using the lemma and Theorem 2, we can write directly the following theorem for the nth eigenvalue derivative that does not require the left eigenvector.

Theorem 3. For the nonlinear eigenvalue problem, the nth derivative of the unrepeated eigenvalue λ is

$$\lambda^{(n)} = -\frac{x^\dagger \mathbf{K} z_n}{x^\dagger \mathbf{K} A_\lambda x} \tag{A12}$$

where $\mathbf{K} = KQ$, $Q = (A + xx^\dagger K)^{-1}$ and $x^\dagger \mathbf{K} A_\lambda x \neq 0$.

Proof. By substituting solution (23) for $x^{(n)}$ into Eq. (17) one gets

$$x^\dagger K\left[-\frac{1}{2}(\alpha\lambda^{(n)} + \eta_n)x - Q(A_\lambda x \lambda^{(n)} + z_n)\right] = -\frac{\alpha\lambda^{(n)}}{2} - \frac{\eta_n}{2}$$

which becomes

$$-\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n) - x^\dagger KQA_\lambda x\lambda^{(n)} - x^\dagger KQz_n = -\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n)$$

or

$$\lambda^{(n)} = -\frac{x^\dagger \mathbf{K} z_n}{x^\dagger \mathbf{K} A_\lambda x}$$

Similarly, Eq. (5) premultiplied by $x^\dagger KQ$ gives

$$x^\dagger KQAx^{(n)} = x^\dagger KQA_\lambda x\left(-\frac{x^\dagger KQ\, z_n}{x^\dagger KQ\, A_\lambda x}\right) - x^\dagger KQz_n$$
$$= 0$$

However, the left-hand side of the equation is also 0 because of the lemma

$$x^\dagger KQAx^{(n)} = x^\dagger A^\dagger Q^\dagger Kx^{(n)}$$
$$= 0$$

If A is K Hermitian then, according to the lemma, $KQ = Q^\dagger K$ and this theorem reduces to Theorem 2.

Theorem 4. For the nonlinear eigenvalue problem, the nth derivative of the eigenvector x is

$$x^{(n)} = -\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n)x - Q(A_\lambda x\lambda^{(n)} + z_n) \tag{A13}$$

where

$$Q = (A + xx^\dagger K)^{-1} \tag{A14}$$

Proof: We rewrite Eqs. (5) and (17) as

$$Ax^{(n)} = -A_\lambda x\lambda^{(n)} - z_n \tag{A15}$$

$$x^\dagger KX^{(n)} = -\frac{\alpha\lambda^{(n)}}{2} - \frac{\eta_n}{2} \tag{A16}$$

respectively. After premultiplying Eq. (A16) by x and adding the resulting equation to Eq. A15, we obtain

$$(A + xx^\dagger K)x^{(n)} = -\left[\lambda^{(n)}\left(\frac{\alpha}{2}I + A_\lambda\right)x + z_n + \frac{\eta_n}{2}x\right]$$

or

$$x^{(n)} = -\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n)x - Q(A_\lambda x\lambda^{(n)} + z_n)$$

which completes the proof.

To verify this solution, $x^{(n)}$ is written as

$$x^{(n)} = -\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n)x - QAx^{(n)}$$

where Eq. (5) was used, which becomes

$$(I - QA)x^{(n)} = -\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n)x$$

However,

$$I - QA = xx^\dagger K$$

because

$$QQ^{-1} = Q(A + xx^\dagger K)$$
$$= I$$

Therefore,

$$xx^\dagger Kx^{(n)} = -\tfrac{1}{2}(\alpha\lambda^{(n)} + \eta_n)x$$

which premultiplied by $x^\dagger K$ gives Eq. (17). This completes the proof.

Fundamental Eqs. (5) and (17) can also be written as a system of matrix equations

$$\begin{pmatrix} x^{(n)} \\ \lambda^{(n)} \end{pmatrix} = \begin{bmatrix} A & A_\lambda x \\ 2x^\dagger K & \alpha \end{bmatrix}^{-1} \begin{pmatrix} -z_n \\ -\eta_n \end{pmatrix} \tag{A17}$$

This block matrix can only be inverted for $\alpha \neq 0$, where $\alpha = x^\dagger K_\lambda x$ is given by Eq. (8). Finding this inverse is somewhat involved and is not elaborated here. The solution, which can easily be verified, is

$$\begin{pmatrix} x^{(n)} \\ \lambda^{(n)} \end{pmatrix} = \begin{bmatrix} P & \dfrac{x}{2} \\ -\dfrac{2}{\alpha}x^\dagger KP & 0 \end{bmatrix} \begin{pmatrix} -z_n \\ -\eta_n \end{pmatrix} \tag{A18}$$

where the $n \times n$ matrix

$$P = \left[A - \frac{2}{\alpha}A_\lambda xx^\dagger K\right]^{-1} \tag{A19}$$

must be nonsingular; this is true if and only if $A_\lambda x \neq 0$ for nontrivial case for which $x \neq 0$. Indeed, from Eq. (A19), it follows that

$$P^{-1}x = -\frac{2}{\alpha}A_\lambda x \tag{A20}$$

and, if P^{-1} is singular ($\det P^{-1} = 0$), then there is a nontrivial solution of the system of equations

$$P^{-1}x = 0$$

which contradicts Eq. (A20). Therefore, $\det P^{-1} \neq 0$, and because $\det P^{-1} \det P = 1$, it must be that $\det P \neq 0$, which we wanted to prove. This leads us to the following theorem.

Theorem 5. For the nonlinear eigenvalue problem for which $\alpha = x^\dagger K_\lambda x \neq 0$ the nth derivatives of the unrepeated eigenvalue λ and its eigenvector x are

$$\lambda^{(n)} = \frac{2}{\alpha}x^\dagger KPz_n \tag{A21}$$

$$x^{(n)} = -Pz_n - \frac{\eta_n}{2}x \tag{A22}$$

respectively, where matrix Eq. (A19) is assumed to be nonsingular.

Proof. The solutions are obtained directly from Eq. (A18). Note that it is much easier to obtain second- and higher-order derivatives using Theorem 5 than using Theorems 5 and 6 in Ref. 13. This theorem is especially useful for problems in dynamics, which specify the normalization condition for which K is a function of λ. Note that it cannot be used for $K = I$ because in this case $\alpha = 0$.

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References

[1]Bellman, R., *Introduction to Matrix Analysis,* McGraw-Hill, New York, 1970, pp. 60–63.

[2]Kato, T., *Perturbation Theory for Linear Operators,* Springer-Verlag, New York, 1984.

[3]Jacobi, C. G. J., "Uber ein leichtes Verfahren die in der Theorie der saecularstoerungen vorkommenden Gleichungen numerish aufzuloesen," *Zeitsschrift für Rein unngewandte Mathematik,* Vol. 30, 1946, pp. 51–95.

[4]Lu, Y., and Murthy, V. R., "Stability Sensitivity Studies for Synthesis of Aeroelastic Systems," *Journal of Aircraft,* Vol. 27, No. 10, 1990, pp. 849, 850.

[5]Jankovic, M. S., "Deployment Dynamics of Flexible Spacecrafts," Ph.D. Thesis, University of Toronto, Toronto, Canada, UTIAS, 1979.

[6]Sobel, K. M., Yu, W., and Lallman, F. J., "Eigenstructure Assignment with Gain Suppression Using Eigenvalue and Eigenvector Derivatives," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 6, 1990, pp. 1008–1013.

[7]Slater, G. L., and McLaren, M. D., "Estimator Eigenvalue Placement in Positive Real Control," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 1, 1990, pp. 168–175.

[8]Karpel, M., "Sensitivity Derivatives of Flutter Characteristics and Stability Margins for Aeroservoelastic Design," *Journal of Aircraft,* Vol. 27, No. 4, 1990, pp. 368–375.

[9]Lim, J. W., "Aeroelastic Optimization of a Helicopter Rotor Using an Efficient Sensitivy Analysis," *Journal of Aircraft,* Vol. 28, No. 1, 1991, pp. 29–37.

[10]Brandon, J. A., "Second-Order Sensitivities to Assess the Applicability of Sensitivity Analysis," *AIAA Journal*, Vol. 29, No. 1, 1991, pp. 135–139.

[11]Bindolino, G., and Mantegazza, P., "Aeroelastic Derivatives as a Sensitivity Analysis of Nonlinear Equations," *AIAA Journal,* Vol. 25, No. 8, 1987, pp. 1145, 1146.

[12]Muthy, D. V., and Kaza, K. R. V., "A Computational Procedure for Automated Flutter Analysis," *Communications in Applied Numerical Methods,* Vol. 5, No. 1, 1989, pp. 29–37.

[13]Jankovic, M. S., "Analytical Solutions for the *n*th Derivatives of Eigenvalues and Eigenvectors for a Nonlinear Eigenvalue Problem," *AIAA Journal*, Vol. 26, No. 2, 1988, pp. 204, 205.

[14]Andrew, A. L., and Chu, K. W. E., "Comment on 'Analytical Solutions for the *n*th Derivatives of Eigenvalues and Eigenvectors for a Nonlinear Eigenvalue Problem'," *AIAA Journal*, Vol. 29, No. 7, 1991, p. 1182.

[15]Jankovic, M. S., "Reply by the Author to A. L. Andrew and K. W. E. Chu," *AIAA Journal,* Vol. 29, No. 7, 1991, pp. 1183,1184.

[16]Chen, L. W., and Ku, D. M., "Eigenvalue Sensitivity in the Stability Analysis of Beck's Column with a Concentrated Mass at the Free End," *Journal of Sound and Vibration,* Vol. 153, No. 3, 1992, pp. 403–411.

[17]Murthy, D. V., and Haftka, R. T., "Derivatives of Eigenvalues and Eigenvectors of a General Complex Matrix," *International Journal for Numerical Methods in Engineering,* Vol. 26, 1988, pp. 293–311.

[18]Adelman, H. M., and Haftka, R. T., "Sensitivity of Discrete Structural Systems," *AIAA Journal,* Vol. 24, No. 5, 1986, pp. 823–832.

[19]Greensite, A. L., *Elements of Modern Control Theory,* Spartan Books, 1970, pp. 396–398.

[20]Fairwether, G., "On the Eigenvalues and Eigenvectors of a Class of Hassenberg Matrices," *SIAM Review,* Vol. 13, No. 2, 1971.

[21]Tan, R. C. E., "Accelerating the Convergence of an Iterative Method for Derivatives of Eigensystems," *Journal of Computational Physics*, Vol. 67, 1986, pp. 230–235.

[22]Tan, R. C. E., "Some Acceleration Methods for Iterative Computation of Derivatives of Eigenvalues and Eigenvectors," *International Journal for Numerical Methods in Engineering,* Vol. 28, 1989, pp. 1505–1519.

[23]Pease, M. C., III, *Methods of Matrix Algebra*, Academic Press, New York, 1965, pp. 247–249.

[24]MATLAB, The MathWorks Inc., Natick, MA.